Differentiability

In one variable

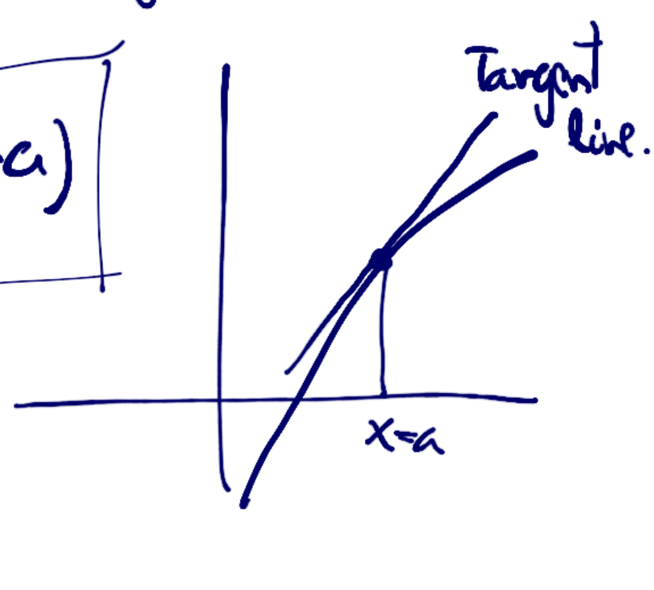
$$f : \mathbb{R} \to \mathbb{R}$$

the derivative at one point $a \in \mathbb{R}$ provides us with ratio of change of the function f at the point $x=a$

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a) = \frac{df(a)}{dx}$$

Geometrically, we have a tangent line at $x=a$

$$\boxed{y = f(a) + f'(a)(x-a)}$$



$f'(a) \equiv$ slope of tangent line

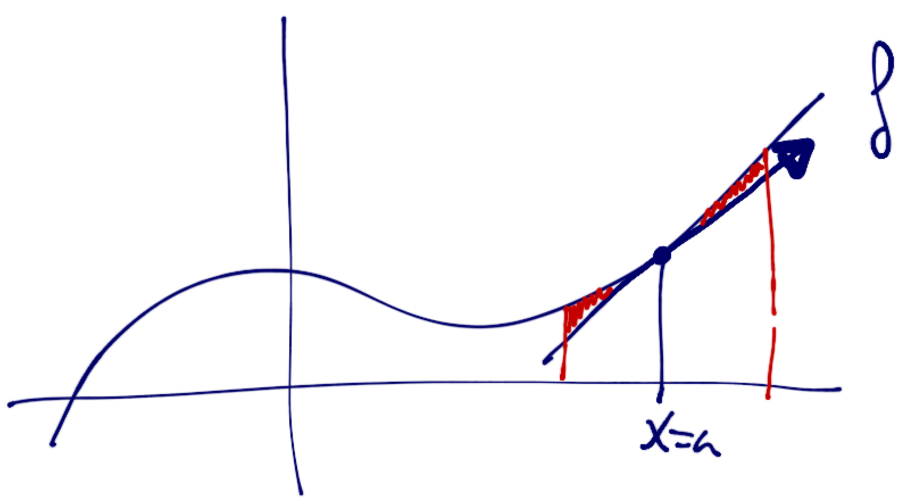
If we are very close to $(a, f(a))$

$$\boxed{f(x) = \underbrace{\overbrace{f(a) + f'(a)(x-a)}^{\text{tangent line}}}_{\text{approximation}} + r(h)}$$

approximation $\quad h \sim 0 \quad$ (Taylor's polyno)

$h = |x-a|$

$r(h) \equiv$ distance between the curve $f(x)$ and the tangent line.



If $r(h) \longrightarrow 0$ as $h \longrightarrow 0$ or $x \longrightarrow a$

we find the continuity of the function f.

For differentiability we need something else.

In fact, we need that

$$\lim_{h \to 0} \frac{r(h)}{h} = 0$$

This guarantees the diff.

Indeed,

$$f(x) - f(a) = f'(a)(x-a) + r(|x-a|)$$

$$\Downarrow$$

$$\frac{f(x)-f(a)}{x-a} = f'(a) + \frac{r(|x-a|)}{x-a}$$

If $x = a+h$

$$\frac{f(a+h)-f(a)}{h} = f'(a) + \frac{r(h)}{h}$$

$h \to 0$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$f'(a) \qquad\quad f'(a) \qquad\quad 0$$

If f is diff.

In one variable

differentiability $\equiv$ existence of derivatives.

We want to generalise these ideas to several variables.

For example

$$z = f(x, y) \quad , \quad f : \mathbb{R}^2 \longrightarrow \mathbb{R}.$$

First, we need to understand the ratio of change for each variable. (derivative)

- We perform a derivative of the function with respect to each variable.

partial derivatives

Definition

Let $f: A \subset \mathbb{R}^n \to \mathbb{R}$ be a function and $x_0 \in A$

Then, we define the partial derivative of f with respect to the variable x_i as

$$\frac{\partial f(x_0)}{\partial x_i} = \lim_{t \to 0} \frac{f(x_0 + te_i) - f(x_0)}{t}$$

$$= \lim_{t \to 0} \frac{f(x_1, \dots, x_i + t, \dots x_n) - f(x_1, \dots x_n)}{t}$$

$e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow$ position i , $e_i \equiv$ vectors of the standard basis

If $f: \mathbb{R}^n \to \mathbb{R}^m$, $f(x) = (f_1(x), \dots, f_m(x))$, $\frac{\partial f_j}{\partial x_i}$ partial derivatives of the j-component with respect to i-variable

Example: • $f(x,y) = xy + x - y$ at $(0,0)$, $\dfrac{\partial f(0,0)}{\partial x}$ $\dfrac{\partial f(0,0)}{\partial y}$?

$$\frac{\partial f(0,0)}{\partial x} = \lim_{h \to 0} \frac{f(0+h,0) - \overbrace{f(0,0)}^{=0}}{h} = \lim_{h \to 0} \frac{\overbrace{h \cdot 0}^{=0} + h - \overbrace{0 - 0}^{=0}}{h}$$

$$= 1$$

$$\frac{\partial f(0,0)}{\partial y} = \lim_{k \to 0} \frac{f(0, \overbrace{k}^{0+k}) - f(0,0)}{k} = \lim_{k \to 0} \frac{-k}{k} = -1$$

• $f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ $\leftarrow f(0,0)=0$

$$\frac{\partial f(0,0)}{\partial x} = \lim_{h \to 0} \frac{f(0+h,0) - \overbrace{f(0,0)}^{=0}}{h} = \lim_{h \to 0} \frac{\frac{0}{h^2+0}}{h} = 0$$

$$\frac{\partial f(0,0)}{\partial y} = \lim_{k \to 0} \frac{f(0,0+k) - f(0,0)}{k} = \lim_{k \to 0} \frac{\left(\frac{0 \cdot k^2}{0+k^2}\right)^{=0}}{k} = 0$$

$$f(x,y) = xy + x - y$$

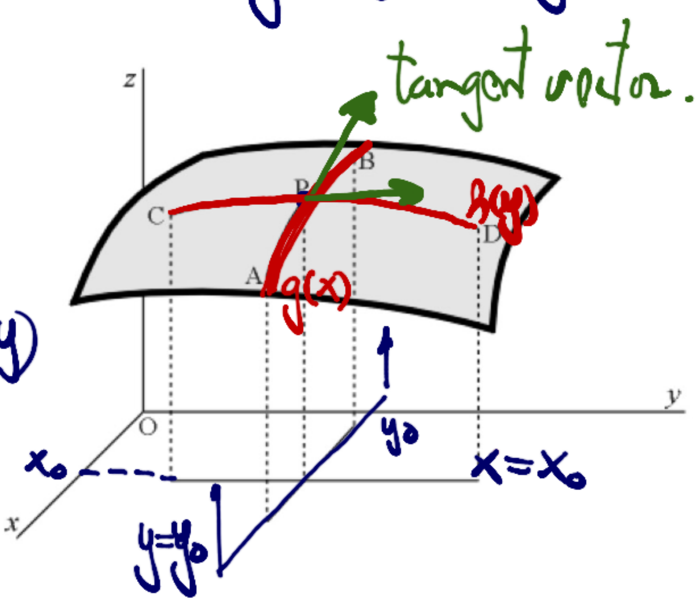
$$\underbrace{\frac{\partial f(x,y)}{\partial x} = y+1}_{\Downarrow} \qquad , \qquad \underbrace{\frac{\partial f(x,y)}{\partial y} = x-1}_{\Downarrow}$$

$$\frac{\partial f(0,0)}{\partial x} = 1 \qquad \frac{\partial f(0,0)}{\partial y} = -1$$

Geometric interpretation

Assume $z = f(x,y)$ and we fix one of the variables.

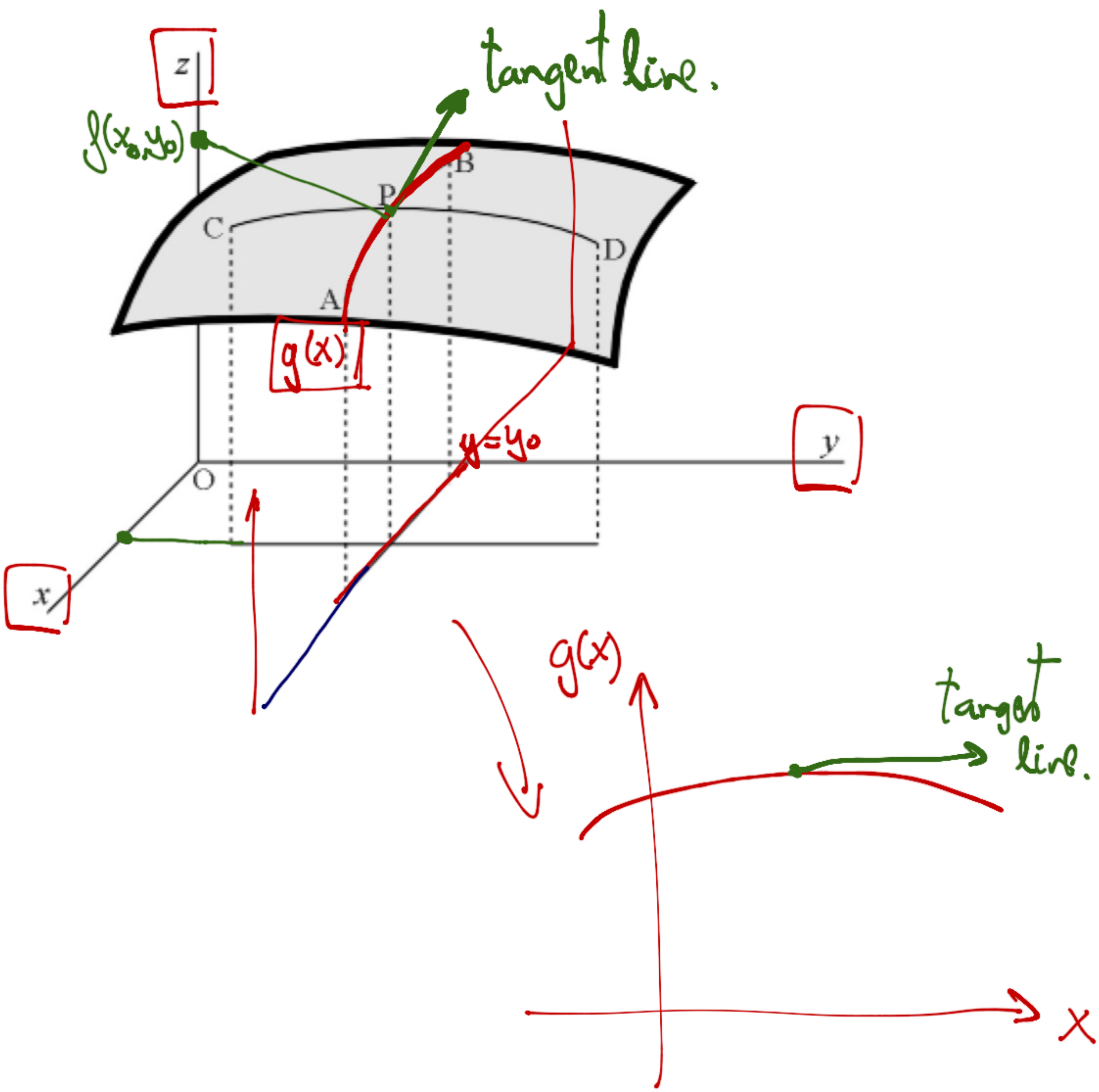
Vertical plane $\left.\begin{array}{l} y = y_0 \\ z = f(x,y) \end{array}\right\} \Rightarrow z = f(x, y_0) = g(x)$



Also,

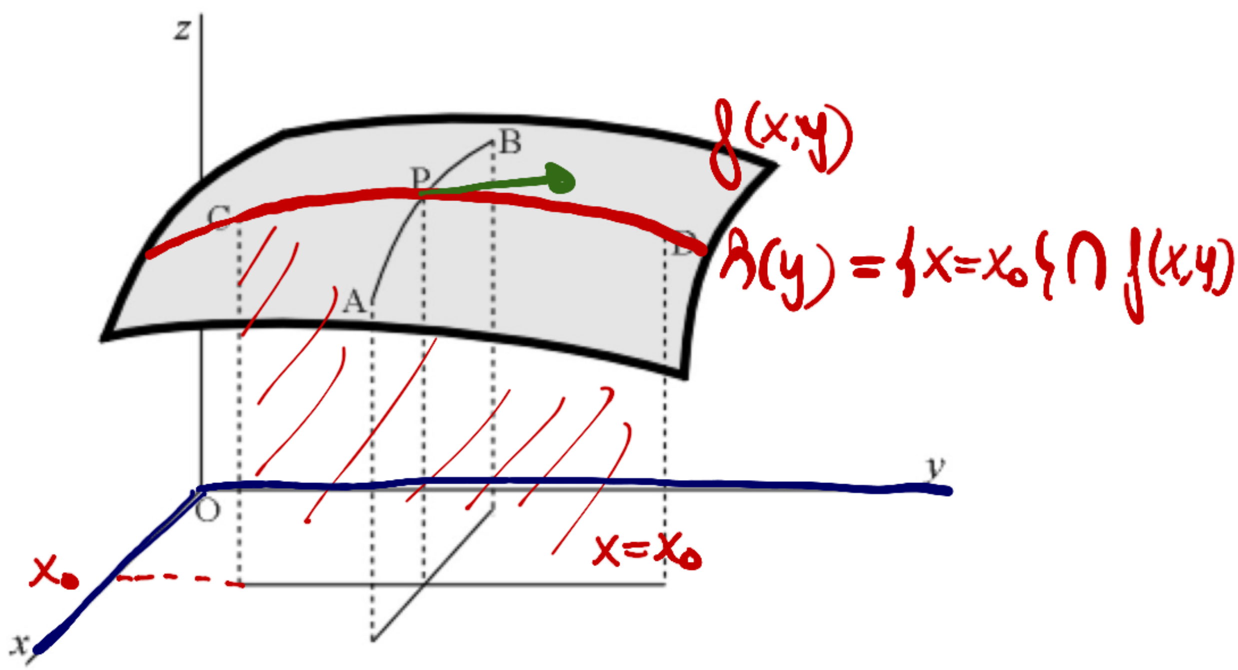
$\left.\begin{array}{l} x = x_0 \\ z = f(x,y) \end{array}\right\} \Rightarrow z = f(x_0, y) = h(y)$

The partial derivative of f with respect to x at P will be represented by the tangent line to the curve $g(x) = f(x, y_0)$



Similarly, the partial derivative of f with respect to y at P will be the tangent line at to the graph of $h(y)$

$$\left.\begin{array}{l} z = f(x,y) \\ x = x_0 \end{array}\right\} \Rightarrow z = f(x_0, y) = h(y)$$



Partial derivatives are the ratio of change for f at P in the directions of X and Y.

We can extend those ideas to any direction.

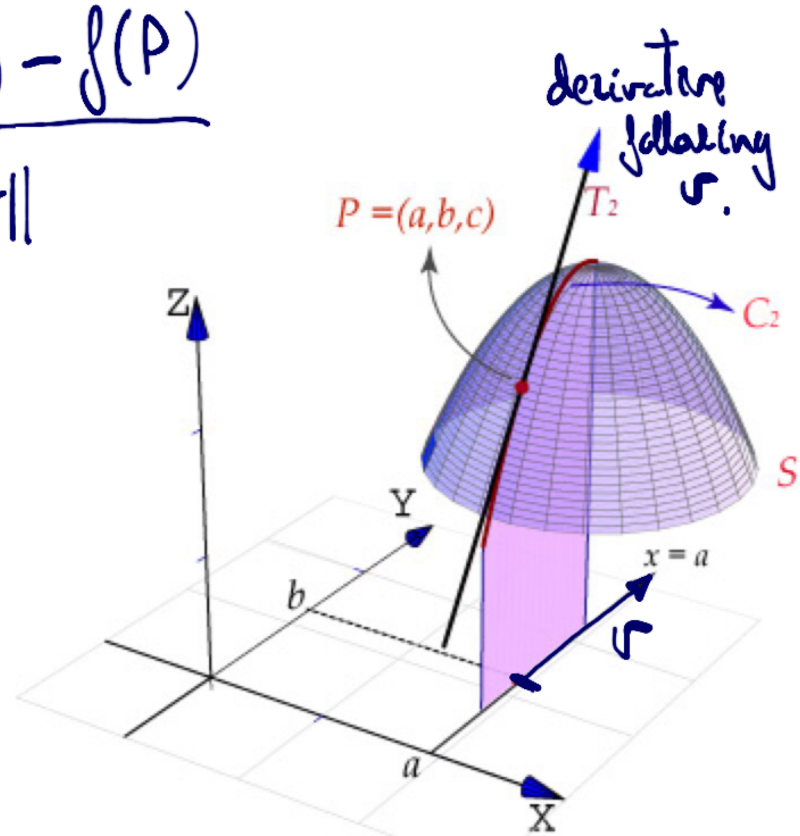
<u>Definition</u> – Directional derivatives

$f: A \subset \mathbb{R}^n \to \mathbb{R}$, $P \in A \subset \mathbb{R}^n$ a point in dom f.

and $v \in \mathbb{R}^n \setminus \{0\}$ any vector.

Then, we say that f has derivative at P on the direction of v when the following limit exists.

$$\lim_{t \to 0} \frac{f(P+tv) - f(P)}{t\|v\|}$$



Remarks

- The concept of directional derivative generalises the concept of partial derivatives to any direction

$$D_v f(P) = \lim_{t \to 0} \frac{f(P+tv) - f(P)}{t\|v\|} = \frac{d}{dt} f(P+tv)\Big|_{t=0}$$

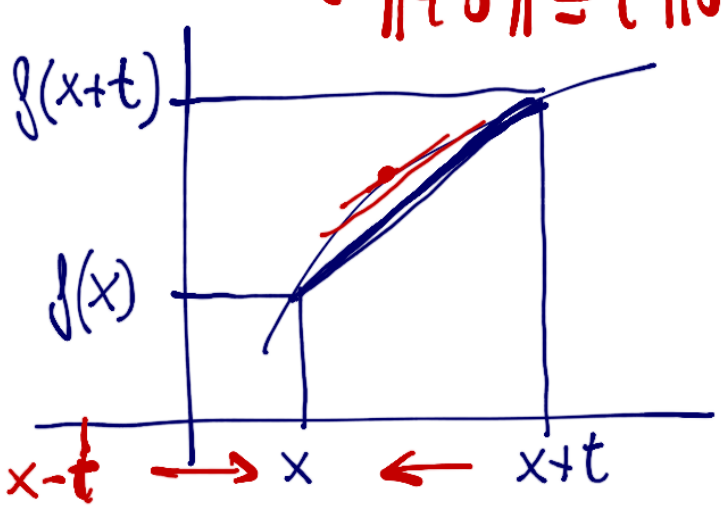
$f : \mathbb{R}^n \to \mathbb{R}$.

$f(P) \in \mathbb{R}$, $P \in \mathbb{R}^n$

How f changes $\to$ $f(P+tv) - f(P)$

in relation to P and $P+tv$ $\to$ $\underbrace{\|P+tv-P\|}_{=\|tv\| = t\|v\|} = dist(P, P+tv)$

In 1D



$$\frac{f(x+t) - f(x)}{x+t-x} = \text{ratio of change.}$$

Examples: $f(x,y)=\sqrt{|xy|}$

Compute the derivative at $(0,0)$ in the direction of $v=(1,1)$

$$D_v f(0,0)=\lim_{t\to 0}\frac{f(t,t)-f(0,0)}{t\,\|(1,1)\|}=$$

$$=\frac{1}{\sqrt{2}}\lim_{t\to 0}\frac{\sqrt{t^2}}{t}=\frac{1}{\sqrt{2}}\lim_{t\to 0}\frac{|t|}{t}$$

The limit does not exist, so the directional derivative does not exist. for $v=(1,1)$

Also,

$$\left.\begin{aligned}\frac{\partial f(0,0)}{\partial x}&=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}=0\\ \frac{\partial f(0,0)}{\partial y}&=\lim_{k\to 0}\frac{f(0,k)-f(0,0)}{k}=0\end{aligned}\right\}$$

Remark

We observe that $\boxed{\text{the existence of all directional derivatives does not guarantee continuity}}$

In 1D derivatives $\Rightarrow$ continuity.

In several dimensions

$\boxed{\text{all directional derivatives} \not\Rightarrow \text{continuity}}$

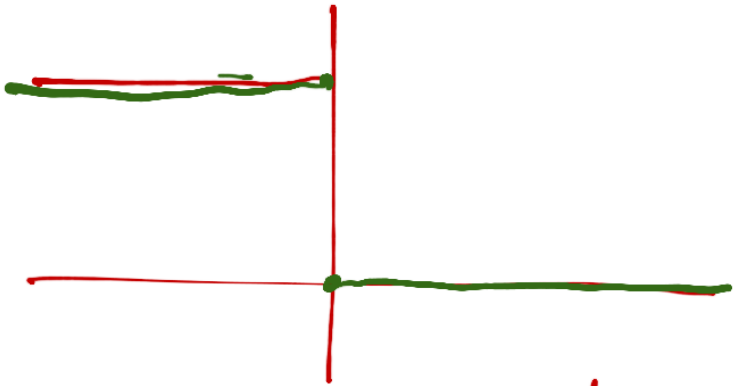
(we are actually approaching following lines)

Example:

$$f(x,y) = \begin{cases} 1 & \text{if } x=0 \text{ or } y=0 \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{\partial f(0,0)}{\partial x} = 0 = \frac{\partial f(0,0)}{\partial y}.$$

But $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist !!!



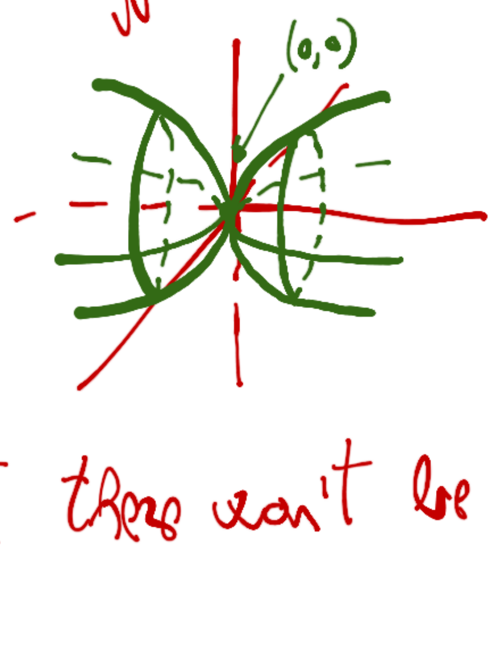
Example: $f(x,y) = x^{1/3}\, y^{1/3}$

$$\frac{\partial f(0,0)}{\partial x} = \frac{\partial f(0,0)}{\partial y} = 0$$

We must apply the definition since the $x^{1/3}$ and $y^{1/3}$ are not diff.

$$\frac{\partial f(x,y)}{\partial x} = \frac{1}{3}\, x^{-2/3}\, y^{1/3}$$

$$\frac{\partial f(x,y)}{\partial y} = \frac{1}{3}\, x^{1/3}\, y^{-2/3}$$



The problem we have is that there won't be a tangent plane.

It looks necessary to have some kind of condition for the derivatives that implies, at least, continuity and also differentiability

diff. $\sim$ having a tangent plane.

* (existence of derivatives in N-dimensions)